The required curves are, then, the central conics:—ellipses if the normal meets the two axes on the same side of the curve; and hyperbolas, if on opposite sides of the curve. The given property might be taken to define the central conic sections.

Also solved by G. B. M. Zerr.

CALCULUS.

254. Proposed by H. S. PARDEE, Boston, Mass.

A wire is wound in the form of a helix. Assuming that sections of the wire perpendicular to the axis of the wire are circles, find the equation of a section of the wire perpendicular to the axis of the helix.

Solution by DR. E. SWIFT, Princeton, University.

The shape of the wire after bending may be changed or distorted. The simplest assumption is that after bending, a section perpendicular to the helix is still a circle, say, of radius R. In this case, the surface of the wire may be regarded as the envelope of a family of spheres, whose centers are on the wire and whose radii are all equal to R. We may write the equation of the helix as

$$\left.\begin{array}{l} x=a\cos t \\ y=a\sin t \\ z=ct \end{array}\right\}$$

where t is parameter, a and c constants depending on the shape and size of the helix. The family of spheres has then for its equation,

$$(x-a\cos t)^2+(y-a\sin t)^2+(z-ct)^2=R^2. \tag{1}$$

To find the envelope we differentiate (1) with respect to t and eliminate t from (1) and the resulting equation. To find the equation of the section by a plane perpendicular to the axis, here the XY-plane, we must set $z=0$. If we do this before eliminating t, we obtain the equation of the plane section in the parametric form

$$\left.\begin{array}{l} (x-a\cos t)^2+(y-a\sin t)^2+(-ct)^2=R^2 \\ a\sin t(x-a\cos t)-a\cos t(y-a\sin t)+c^2t=0 \end{array}\right\} \tag{2}$$

and these equations show that the section is *the envelope of circles whose centers lie on the circle* $\left.\begin{array}{l} x=a\cos t \\ y=a\sin t \end{array}\right\}$ *and whose radii are* $\sqrt{(R^2-c^2t^2)}$.

It is possible to solve these equations for x and y in terms of t. Expanding them, we have

$$\left.\begin{array}{l} x^2-2ax\cos t+y^2-2ay\sin t+a^2+c^2t^2-R^2=0 \\ ax\sin t-ay\cos t+c^2t=0 \end{array}\right\} \qquad (3)$$

or

$$\left.\begin{array}{l} (x^2+y^2)+a^2+c^2t^2-R^2=2a(x\cos t+y\sin t) \\ 2c^2t=2a(-x\sin t+y\cos t) \end{array}\right\}. \qquad (4)$$

Squaring and adding, we have

$$(x^2+y^2)^2+2(x^2+y^2)(a^2+c^2t^2-R^2)+(a^2+c^2t^2-R^2)^2+4c^4t^2=4a^2(x^2+y^2)$$

a quadratic equation for x^2+y^2, or the square of the radius vector r. And if we use polar coordinates, $\left.\begin{array}{l} x=r\cos\phi \\ y=r\sin\phi \end{array}\right\}$, this equation enables us to find r. Putting for x and y, $r\cos\phi$ and $r\sin\phi$, respectively, in the last of equations (4), we have

$$c^2t=ar\sin(\phi-t), \text{ or } \phi=t+\sin^{-1}\frac{c^2t}{ar}.$$

The equations of the curve are then

$$r=\sqrt{[2\sqrt{(a^2R^2-a^2c^2t^2-c^4t^2)}-(c^2t^2-a^2-R^2)]}, \quad \phi=t+\sin^{-1}\frac{c^2t}{ar}.$$

Also solved by G. B. M. Zerr. An incomplete solution was received from the Proposer.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

145. Proposed by J. D. WILLIAMS, being the 12th of his fourteen challenge problems proposed in 1832.

Make $x^2+y^2=\square$, $\frac{5}{4}(x^2+y^2)=$a cube, $xy=2x^3$, $2(x+y)+\frac{xy}{x+y}=\square$, and $(x^4+y^4)(x^2+y^2)-(x^5+y^5)\sqrt{(x^2+y^2)}=\square$.

Solution by DR. E. SWIFT, Princeton University.

If $x^2+y^2=\square$, we must have $x=2\xi\eta k$, $y=(\xi^2-\eta^2)k$, where ξ, η, k are integers. Then $x^2+y^2=k^2[\xi^2+\eta^2]^2$,

If $\frac{5}{4}$ of this=a cube, evidently it must be of the form $4\times25\times$a cube, or $k(\xi^2+\eta^2)$ is of the form $2\times5\times$a cube.

Since $xy=2x^3$, either, a) $x=0$, or, b) $y=2x^2$.

If a) is true, $y=\sqrt{(x^2+y^2)}$ and must be of the form $2\times5\times$a cube.

$x=0$ satisfies the last condition; it remains to satisfy the third, which reduces to $2y=\square$

Since $y=10a^3$, the least value of a which makes $2y^2$ square is 5, and $y=10\times125=1250$.